NOTES ON OPERADS

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ABSTRACT. This note is for a talk on operads. The main reference is [1]. The books [2, 3] are also useful.

CONTENTS

1. OPERAD

1.1. Tree.

Definition 1.1. *A* ***graph*** $\Gamma = (V_\Gamma, E_\Gamma)$ *is a pair of sets where* E_Γ *is contained in the power set* 2^{V_Γ} *(the set of subsets in* V_Γ*). A* ***directed graph*** *is a graph* $\Gamma = (V_\Gamma, E_\Gamma)$ *with* ***source map*** *and* ***target map*** $s, t : E_\Gamma \to V_\Gamma$ *such that* $e = \{s(e), t(e)\}$ *for any* $e \in E_\Gamma$*. An* ***isomorphism*** $\Phi : \Gamma \to \tilde{\Gamma}$ *of graphs from* $\Gamma = (V_\Gamma, E_\Gamma)$ *to* $\tilde{\Gamma} = (V_{\tilde{\Gamma}}, E_{\tilde{\Gamma}})$ *consists of bijections* $\Phi_V : V_\Gamma \to V_{\tilde{\Gamma}}$ *and* $\Phi_E : E_\Gamma \to E_{\tilde{\Gamma}}$ *such that* $\Phi_E(\{v, w\}) = \{\Phi_V(v), \Phi_V(w)\}$ *for any* $\{v, w\} \in E_\Gamma$*. An* ***isomorphism of directed graphs*** *is an isomorphism of graphs which is compatible with the source and target maps. Let* $v \in V_\Gamma$*. We denote*

$$A(v) := \{e \in E_\Gamma \mid v \in e\}.$$

The number $|A(v)|$ *is called the* ***valency*** *of* v*. An edge* $e \in E_\Gamma$ *is called a* ***cycle*** *if* $|e| = 1$*.*

Definition 1.2. *A* ***tree*** $T = (v_o, V_T, E_T)$ *is a connected graph without cycles which has a special vertex* $v_o \in V_T$*, called* ***root vertex****, such that* $|A(v_o)| = 1$*. The edge adjacent to* v_o *is called the* ***root edge****, denoted* e_o*. Non-root vertexes of valency* 1 *are called* ***leaves****. The set of leaves of* T *is denoted* $L(T)$*. A vertex is called* ***internal*** *if it is neither a root nor a leaf.*

Remark 1.3. *A tree, with the direction towards the root, is naturally a directed graph.*

Definition 1.4. *A tree* T *is called* ***planar*** *if for every internal vertex of* T*, the set* $t^{-1}(v)$ *carries a total order. An* n***-labeled planar tree*** *is a planar tree equipped with an injective map* $\mathfrak{l} : \{1, \cdots, n\} \to L(T)$*. (The map* $\mathfrak{l}$ *is not required to be monotone.) A vertex* v *of an* n*-labeled planar tree* T *is called* ***nodal*** *if* $v \in N_T := V_T \setminus \{v_o\} \setminus \operatorname{im} \mathfrak{l}$*.*

Let S, T *be* n*-labeled planar trees. A* ***(non-planar) morphism*** $\Phi : S \to T$ *is a pair of bijections* $\Phi_V : V_S \to V_T$ *and* $\Phi_E : E_S \to E_T$ *which are compatible with source and target maps, and* $\Phi_V \circ \mathfrak{l}_S = \mathfrak{l}_T$*. The category* *of n-labeled planar trees is denoted* $\mathrm{Tree}(n)$. *The full subcategory of n-labeled planar trees with k nodal vertexes is denoted* $\mathrm{Tree}_k(n)$.

Remark 1.5. *There is a natural left S_n-action on the objects of* $\mathrm{Tree}(n)$.

1.2. **Operad and cooperad.** Let $\mathfrak{C}$ be the category of cochain complexes.

Definition 1.6. *A **S-module** is a sequence $\{P(n)\}_{n\geq 0}$ of objects in $\mathfrak{C}$ such that for each $n \in \mathbb{N}_0$, the object $P(n)$ is equipped with a left S_n-action.*

Let $T \in \mathrm{Tree}(n)$. Define

$$P(T) := \bigotimes_{v \in N_T} P(|t^{-1}(v)|)$$

where the tensor product is done in the order induced by T.

Definition 1.7. *A **(dg) operad** is an S-module $\{P(n)\}_{n\geq 0}$ equipped with "composition maps"*

$$\mu_T : P(T) \to P(n)$$

*for any $T \in \mathrm{Tree}(n)$, and equipped with a **unit** $u : \Bbbk \to P(1)$ which satisfies a list of axioms ("associativity," "S-equivalent," "unit").*

Proposition 1.8. *Let V be a cochain complex. The direct sum*

$$P(V) := \bigoplus_{n=0}^{\infty} \left(P(n) \otimes V^{\otimes n}\right)_{S_n}$$

with the natural P-algebra structure is the free P-algebra generated by V.

Consider the S-module

$$\Lambda(n) := \begin{cases} s^{1-n}\,\mathrm{sign}_n, & n \geq 1; \\ 0, & n = 0, \end{cases}$$

where $\mathrm{sign}_n = \Bbbk$ with the S_n-action $\sigma \cdot 1 := (-1)^{\sigma} \cdot 1$. The compositions are defined by

$$1_m \circ_i 1_n := (-1)^{(1-n)(i-1)} 1_{n+m-1}.$$

Remark 1.9. *The sign assignment of insertion is different from* [1]. *It is not clear to the author how the sign convention was chosen in* [1].

Let V be a cochain complex, and let

$$\Phi : \Lambda \to \mathrm{End}_V$$

be a morphism of dg operads. Let $\tilde{\Phi} : \mathrm{Com} \to \mathrm{End}_{V[1]}$ be the map

$$\tilde{\Phi}(\tilde{1}_n)(v_1, \cdots, v_n) := (-1)^{\sum_{j=1}^{n}(n-j)|v_j|}\, s^{-1} \circ \Phi(1_n)(sv_1, \cdots, sv_n).$$

Proposition 1.10. *The assignment*

$$\Lambda\text{-Alg} \to \mathrm{Com}\text{-}\mathrm{Alg}_1 : \Phi \mapsto \tilde{\Phi}$$

is a bijection, where $V \in \mathrm{Com}\text{-}\mathrm{Alg}_1$ iff $V[1] \in \mathrm{Com}\text{-}\mathrm{Alg}$.

Proof. We prove $\tilde{\Phi}$ is a morphism of operads. The other parts of proof should be easy. Since Φ is a morphism, we have

$$\Phi(1_n)(v_{\sigma(1)}, \cdots, v_{\sigma(n)}) = \epsilon(\sigma, v)(-1)^{\sigma}\, \Phi(1_n)(v_1, \cdots, v_n)$$

Then

$$\begin{aligned}\tilde{\Phi}(\sigma\star\tilde{1}_n)(v_1,\cdots,v_n) &= (-1)^{\sigma}(-1)^{\sum_{j=1}^{n}(n-j)|v_j|}\, s^{-1}\circ\Phi(\sigma\cdot 1_n)(sv_1,\cdots,sv_n)\\ &= (-1)^{\sigma}\epsilon(\sigma,sv)(-1)^{\sum_{j=1}^{n}(n-j)|v_j|}\, s^{-1}\circ\Phi(1_n)(sv_{\sigma(1)},\cdots,sv_{\sigma(n)})\\ &= \epsilon(\sigma,v)(-1)^{\sum_{j=1}^{n}(n-j)|v_j|}\, s^{-1}\circ\Phi(1_n)(sv_{\sigma(1)},\cdots,sv_{\sigma(n)})\\ &= \tilde{\Phi}(\tilde{1}_n)\big(\sigma\star(v_1\otimes\cdots v_n)\big).\end{aligned}$$

and

$$\begin{aligned}&\tilde{\Phi}(\tilde{1}_m\bar{\circ}_i\tilde{1}_n)(v_1,\cdots,v_{m+n-1})\\ &= (-1)^{\sum_{j=1}^{m+n-1}(m+n-1-j)|v_j|}\, s^{-1}\circ\Phi(1_{m+n-1})(sv_1,\cdots,sv_{m+n-1})\\ &= (-1)^{(1-n)(i-1)}(-1)^{\sum_{j=1}^{m+n-1}(m+n-1-j)|v_j|}\, s^{-1}\circ\Phi(1_m\circ_i 1_n)(sv_1,\cdots,sv_{m+n-1})\\ &= (-1)^{(1-n)(i-1)}(-1)^{\sum_{j=1}^{m+n-1}(m+n-1-j)|v_j|}(-1)^{|\Phi(1_n)|(i-1+\sum_{j=1}^{i-1}|v_j|)}\\ &\qquad\cdot s^{-1}\circ\Phi(1_m)\big(sv_1,\cdots,sv_{i-1},ss^{-1}\Phi(1_n)(sv_i,\cdots,sv_{i+n-1}),sv_{i+n},\cdots,sv_{m+n-1}\big)\\ &= (-1)^{(1-n)(i-1)}(-1)^{\sum_{j=1}^{m+n-1}(m+n-1-j)|v_j|}(-1)^{(1-n)(i-1+\sum_{j=1}^{i-1}|v_j|)}\\ &\qquad\cdot(-1)^{\sum_{j=i}^{i+n-1}(n+i-1-j)|v_j|}(-1)^{\sum_{j=1}^{i-1}(m-j)|v_j|}(-1)^{\sum_{j=i+n}^{m+n-1}(m+n-1-j)|v_j|}(-1)^{\sum_{j=i}^{i+n-1}(m-i)|v_j|}\\ &\qquad\qquad\cdot\tilde{\Phi}(\tilde{1}_m)\big(v_1,\cdots,v_{i-1},\tilde{\Phi}(\tilde{1}_n)(v_i,\cdots,v_{i+n-1}),v_{i+n},\cdots,v_{m+n-1}\big)\\ &= \tilde{\Phi}(\tilde{1}_m)\big(v_1,\cdots,v_{i-1},\tilde{\Phi}(\tilde{1}_n)(v_i,\cdots,v_{i+n-1}),v_{i+n},\cdots,v_{m+n-1}\big)\\ &= \big(\tilde{\Phi}(\tilde{1}_m)\bar{\circ}_i\tilde{\Phi}(\tilde{1}_n)\big)\big(v_1,\cdots,v_{m+n-1}\big).\end{aligned}$$

□

Definition 1.11. *A* ***(dg) cooperad*** *is an S-module $\{Q(n)\}_{n\geq 0}$ equipped with "decomposition maps"*

$$\Delta_T : Q(n)\to Q(T)$$

for any $T\in\mathrm{Tree}(n)$, and equipped with a ***counit*** *$\tilde{u}: Q(1)\to\Bbbk$ which satisfies a list of axioms ("coassociativity," "S-equivalent," "counit").*

A cooperad Q is ***coaugmented*** *if we have a cooperad morphism $\epsilon : *\to Q$, where $*$ is the natural cooparad with $*(1)=\Bbbk$ and $*(n)=0$ if $n\neq 1$.*

We denote the pseudo-cooperad $\mathrm{coker}(\epsilon)$ by Q_o.

Example 1.12. *The S-module Λ also caries a cooperad structure:*

$$\Delta_i : \Lambda_{m+n-1}\to\Lambda_m\otimes\Lambda_n,$$
$$\Delta_i(1_{m+n-1}) := (-1)^{(1-n)(i-1)}\cdot 1_m\otimes 1_n.$$

2. Convolution Lie algebra

The notation π_0 denotes the collection of isomorphism classes in a category.

Let P be a dg (pseudo-)operad, and Q be a dg (pseudo-)cooperad. Consider

$$\mathrm{Conv}(Q,P) := \prod_{n\geq 0}\mathrm{Hom}_{S_n}(Q(n),P(n))$$

with the operation $\bullet$ defined by the sum of the compositions

$$Q(n)\xrightarrow{\Delta_T}Q(n_1)\otimes Q(n_2)\xrightarrow{f\otimes g}P(n_1)\otimes P(n_2)\xrightarrow{\mu_T}P(n)$$

where $T \in \mathrm{Tree}_2(n)$, $n_i = |t^{-1}(v_i)|$, $N_T = \{v_1, v_2\}$. More precisely,

$$f \bullet g(x) := \sum_{T \in \pi_0(\mathrm{Tree}_2(n))} \mu_T \circ (f \otimes g) \circ \Delta_T(x)$$

for $x \in Q(n)$.

Lemma 2.1. *The bracket*

$$[f, g] := f \bullet g - (-1)^{|f||g|} g \bullet f$$

satisfies the Jacobi identity.

The differentials on P and Q induce a differential on the convolution $\mathrm{Conv}(Q, P)$.

Proposition 2.2. *The convolution* $\mathrm{Conv}(Q, P)$ *is a dgla.*

2.1. **Example: cooperad of cocommutative coalgebras.** Let coCom be the cooperad of cocommutative coassociative coalgebras. More precisely,

$$\mathrm{coCom}(n) := \begin{cases} 0, & n = 0; \\ \Bbbk \cdot \delta^n, & n \neq 0, \end{cases}$$

with trivial S_n-action and with the cocompositions

$$\Delta_T : \mathrm{coCom}(n) \to \mathrm{coCom}(n_1) \otimes \mathrm{coCom}(n_2) : \delta^n \mapsto \delta^{n_1} \otimes \delta^{n_2}$$

for $T \in \mathrm{Tree}_2(n)$. We endow coCom with the coaugmentation $\epsilon : * \to \mathrm{coCom} : 1 \mapsto \delta^0$.

If V is a cochain complex, then $\mathrm{coCom}(V) \cong S^{\geq 1}V$ with the differential induced from V and the natural comultiplication.

Proposition 2.3. *Let V be a cochain complex. Then*

$$\mathrm{Conv}(\mathrm{coCom}_o, \mathrm{End}_V) \cong \mathrm{coDer}'(\mathrm{coCom}(V)),$$

where $\mathrm{coDer}'(\mathrm{coCom}(V))$ *is the set of coderivations on* $\mathrm{coCom}(V) \cong S^{\geq 1}V$ *which vanish on* V.

Proof. Note that

$$\begin{aligned} \mathrm{coDer}'(\mathrm{coCom}(V)) &\cong \mathrm{Hom}(S^{\geq 2}V, V) \\ &\cong \prod_{n=2}^{\infty} \mathrm{Hom}(S^n V, V) \\ &\cong \prod_{n=2}^{\infty} \mathrm{Hom}\left(\Bbbk, \mathrm{Hom}(S^n V, V)\right) \\ &\cong \prod_{n=0}^{\infty} \mathrm{Hom}\left(\mathrm{coCom}_o(n), \mathrm{Hom}(S^n V, V)\right) \\ &\cong \prod_{n=0}^{\infty} \mathrm{Hom}_{S_n}\left(\mathrm{coCom}_o(n), \mathrm{Hom}(V^{\otimes n}, V)\right). \end{aligned}$$

It's straightforward to check the isomorphisms preserve the dgla structures. □

Remark 2.4. *According to* [1]*, the above proposition is true for general coaugmented cooperads.*

2.2. **Cobar construction.** Let Q be a coaugmented dg cooperad. Recall that the cobar operad $\Omega(Q)$ associated to Q is quasi-freely generated by $Q_o[-1]$ with the differentials induced by the differentials of Q.

Let P be a dg operad, and let $F : \Omega(Q) \to P$ be a map of dg operads. The restriction

$$F|_{Q_o[-1]} : Q_o[-1] \to P$$

induces a degree one element

$$\alpha_F \in \mathrm{Conv}(Q_o, P).$$

Proposition 2.5. *The map*

$$\mathrm{Mor}(\Omega(Q), P) \to \mathrm{MC}(\mathrm{Conv}(Q_o, P)) : F \mapsto \alpha_F$$

is a bijection.

Corollary 2.6. *Let V be a cochain complex. The L_∞ structures on V is in bijection with the Maurer–Cartan solutions* $\mathrm{MC}(\mathrm{Conv}(\mathrm{coCom}_o, \mathrm{End}_V)) = \mathrm{MC}\left(\mathrm{coDer}(S^{\geq 1}V)\right)$.

Proof. $L_\infty = \Omega(\Lambda\, \mathrm{coCom}_o)$. ☐

References

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